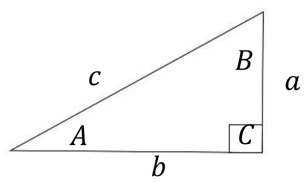
SNAP Centre Workshop

Introduction to Trigonometry

Right Triangle Review

A **right triangle** is any triangle that contains a 90 degree angle.

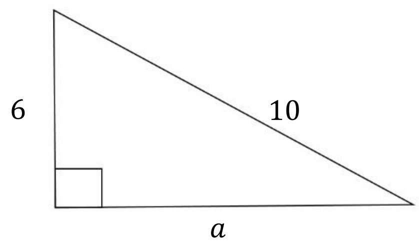
There are six pieces of information we can know about a given right triangle: the lengths of its longest side c (**hypotenuse**) and two shorter sides a and b (**legs**), and the three angles A, B, and C (one of which – by definition – is 90 degrees).



Knowing the length of any two of the sides, the third can be found using the **Pythagorean Theorem**.

Pythagorean Theorem $a^2 + b^2 = c^2$

Example 1



$Find\ the\ length\ of\ a.$

Rearranging the Pythagorean Theorem allows us to solve for the length of a*.*

$$a = \sqrt{c^2 - b^2}$$

$$a = \sqrt{10^2 - 6^2} = \sqrt{100 - 36} = \sqrt{64} = 8$$

The missing length value of our triangle is 8.

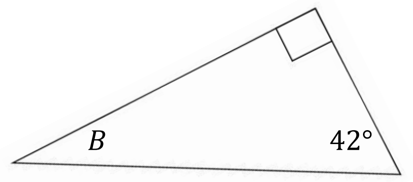
Similarly, since angle C is 90 degrees, and our three angles must sum to 180 degrees, we know that angles A and B must sum to 90 degrees.

$$A + B + C = 180° \qquad Where\ C = 90°$$

$$A + B + 90° = 180°$$

$$A + B = 90°$$

Example 2 *Find the angle B.*



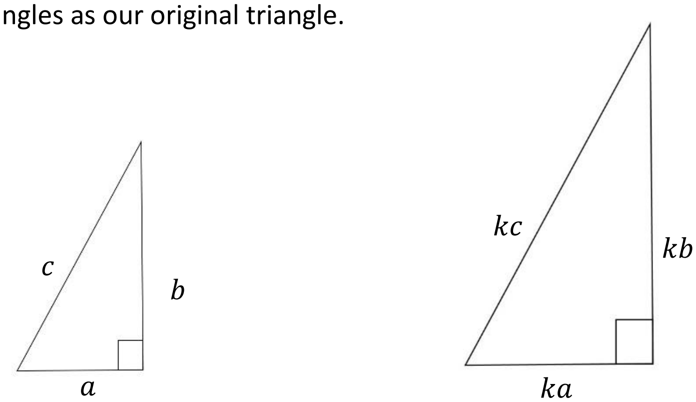
$$B = 90 - A$$

Our value for A, in this case, is $42°$.

$$B = (90 - 42)° = 48°$$

Relating Similar Triangles

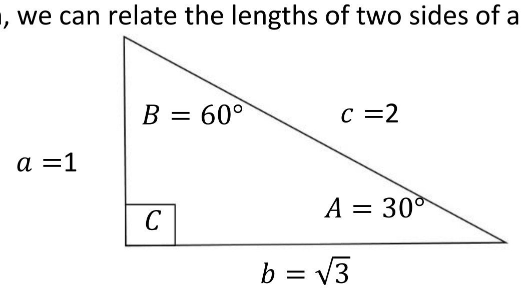
When we multiply the sides of a right triangle by some constant k, we get a triangle of a different size, but with the same angles as our original triangle.



If we take the ratio of sides ka and kb on our larger triangle, we see that the constant coefficients cancel, and the ratio is equal to the ratio of sides a and b on our smaller triangle.

$$\frac{\cancel{k}a}{\cancel{k}b} = \frac{a}{b}$$

Using this information, we can relate the lengths of two sides of a given right triangle to its angles.

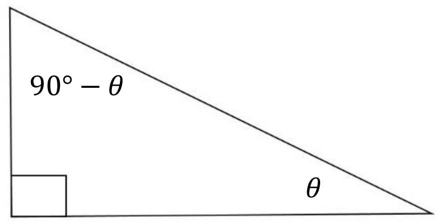


This standard 30-60-90 triangle serves to illustrate the concept of relating the ratios of a right triangle's sides to its angles. If we have a right triangle whose angle A is $30°$, then we know the ratio of sides b to a must be $\frac{\sqrt{3}}{2}$, the ratio of sides a to c must to $\frac{1}{2}$, and b to c must be $\frac{\sqrt{3}}{2}$. Similarly, if we have a right triangle whose sides $a, b, and\ c$ form these ratios, we know angle A **must be** $30°$.

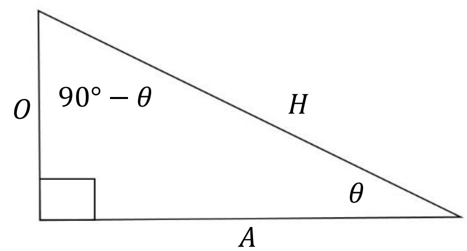
Trigonometric Ratios

When defining the standard **trigonometric ratios**, we first need to decide on an angle to act as a point of reference (traditionally labelled **theta** (θ)). Without a reference angle, we would need to define each angle and side every, single time we study a right triangle.

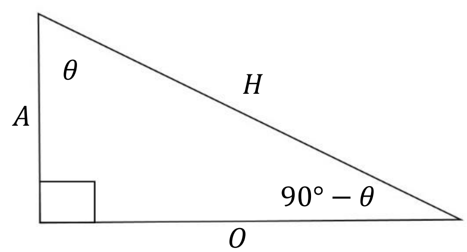
Either of the two angles in our right triangle that are not the 90 degree angle can be theta. The other angle, by default, is $90° - \theta$.



Once we have decided on theta, we label the three sides of our triangle in relation to it. The longest side remains the **hypotenuse** (H), the side across from theta is labelled the **opposite** side (O), and the side that forms theta with the hypotenuse is labelled the **adjacent** side (A).



Note: It is equally valid to choose the other non-90 degree angle to be theta. If we do, it is necessary to label our triangle differently.



Using these new labels, we define the three main trigonometric ratios and their reciprocals: the sine of an angle ($\sin\theta$) is the ratio of the opposite side to the hypotenuse, the cosine of an angle ($\cos\theta$) is the ratio of the adjacent side to the hypotenuse, and the tangent of an angle ($\tan\theta$) is the ratio of the opposite side to the adjacent side.

The reciprocals of sine, cosine, and tangent are cosecant ($\csc\theta$), secant ($\sec\theta$), and cotangent ($\cot\theta$), respectively.

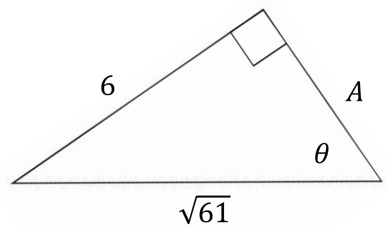
$\sin\theta = \frac{O}{H} = \frac{1}{\csc\theta}$ $\quad$ $\csc\theta = \frac{H}{O} = \frac{1}{\sin\theta}$

$\cos\theta = \frac{A}{H} = \frac{1}{\sec\theta}$ $\quad$ $\sec\theta = \frac{H}{A} = \frac{1}{\cos\theta}$

$\tan\theta = \frac{O}{A} = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta}$ $\quad$ $\cot\theta = \frac{A}{O} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$

So, if we know a non-90 degree angle and any of the three sides of a given right angle, we can use these definitions to solve for the value of any angle or side not given.

Example 3 $\quad$ $Express\ the\ ratios\ \sin\theta, \tan\theta, and\ \sec\theta.$



In order to express $\sin\theta$*, we need to refer to our ratio definitions, then take the values off the right triangle and substitute them into the equation. We finish by rationalizing our fraction.*

$$\sin\theta = \frac{O}{H} = \frac{6}{\sqrt{61}} = \frac{6\sqrt{61}}{61}$$

To express $\tan\theta$*, we need to solve for our adjacent side. We can do this using the Pythagorean Theorem.*

$$A = \sqrt{H^2 - O^2} = \sqrt{\left(\sqrt{61}\right)^2 - 6^2} = \sqrt{61 - 36} = \sqrt{25} = 5$$

$$\tan\theta = \frac{6}{5}$$

Our last ratio to express, $\sec\theta$*, is the inverse of* $\cos\theta$*. With all of the sides of our triangle defined, it can be expressed easily.*

$$\sec\theta = \frac{H}{A} = \frac{\sqrt{61}}{5}$$

Trigonometric Inverses

The inverse trigonometric functions are used when two sides of a right triangle are known, and the goal is to solve for a related angle. The inverse of sine ($\arcsin\left(\frac{O}{H}\right)\ or\ \sin^{-1}\left(\frac{O}{H}\right)$), for example, can be used to solve for theta if the ratio of the opposite side and the hypotenuse are given.

Note: The -1 does not imply $\frac{1}{\sin\left(\frac{O}{H}\right)}$, rather that $\sin^{-1}\left(\frac{O}{H}\right)$ is an inverse trigonometric function.

Each ratio and its reciprocal has a corresponding inverse function:

$$\arcsin\left(\frac{O}{H}\right) = \sin^{-1}\left(\frac{O}{H}\right) = \theta \qquad \operatorname{arccsc}\left(\frac{H}{O}\right) = \csc^{-1}\left(\frac{H}{O}\right) = \theta$$

$$\arccos\left(\frac{A}{H}\right) = \cos^{-1}\left(\frac{A}{H}\right) = \theta \qquad \operatorname{arcsec}\left(\frac{H}{A}\right) = \sec^{-1}\left(\frac{H}{A}\right) = \theta$$

$$\arctan\left(\frac{O}{A}\right) = \tan^{-1}\left(\frac{O}{A}\right) = \theta \qquad \operatorname{arccot}\left(\frac{A}{O}\right) = \cot^{-1}\left(\frac{A}{O}\right) = \theta$$

Example 4 $\quad$ *With a calculator, use inverse ratios to calculate θ from **example 3**.*

We can calculate theta using any of our inverse ratios, resulting in an approximate angle value.

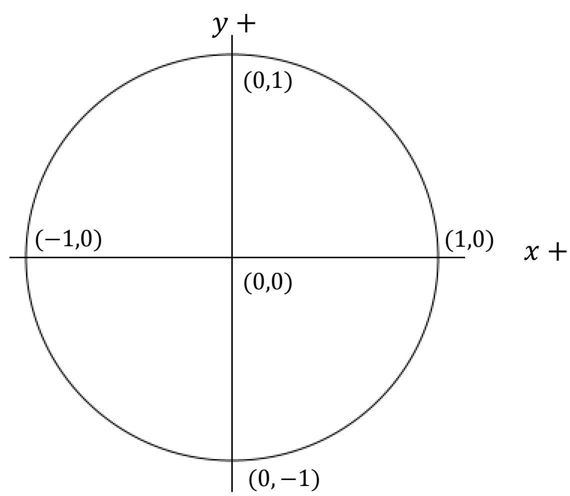
$$\sin^{-1}\left(\frac{6\sqrt{61}}{61}\right) = \sin^{-1}(0.768) = \mathbf{50.2}° = \boldsymbol{\theta}$$

We can check this approximation using one of our other inverse ratios.

$$\tan^{-1}\left(\frac{6}{5}\right) = \tan^{-1}(1.2) = \mathbf{50.2}° = \boldsymbol{\theta}$$

The Unit Circle

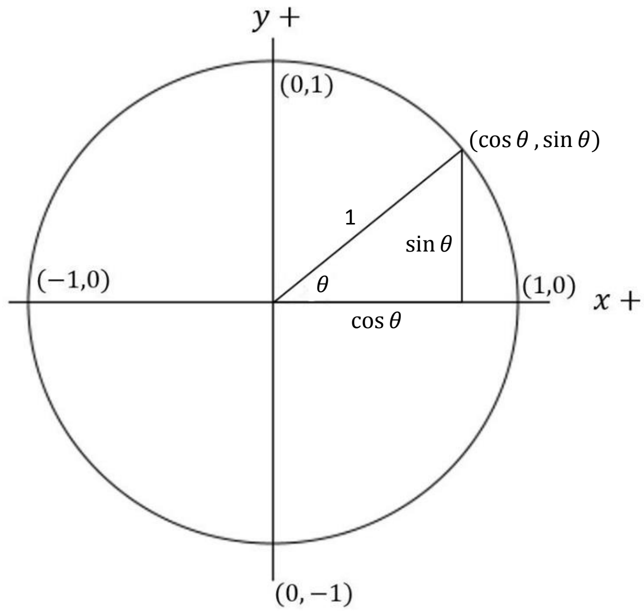
The unit circle is a circle with a radius of 1, centered at the origin of a Cartesian plane.



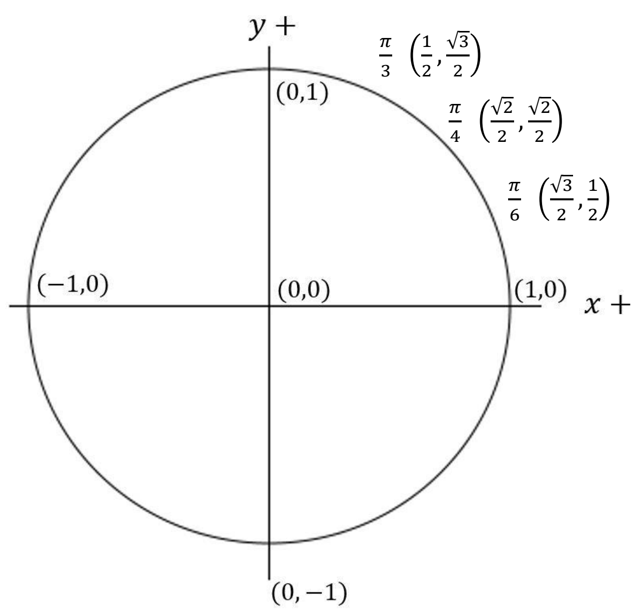
We relate the unit circle to trigonometry by superimposing triangles over it, using the ray from the origin to the circle's perimeter as our hypotenuse, and the angle between this ray and the positive horizontal axis (measured counter-clockwise) as theta.

Note: When discussing the unit circle, we will be expressing angles in terms of radians. The degrees-to-radians conversion factor is $180° = \pi$.

Since $\sin\theta = \frac{Opposite}{Hypotenuse}$, and using the fact that our hypotenuse is 1, we know that $\sin\theta = \frac{Opposite}{1} = Opposite$, or the ***height*** of our superimposed triangle measured against the vertical axis. Likewise, since $\cos\theta = \frac{Adjacent}{Hypotenuse}$, we know that $\cos\theta = Adjacent$, or the ***base*** of our superimposed triangle measured against the horizontal axis. Another way of thinking about this is that any point on the perimeter of the unit circle can be expressed as the (x, y) ordered pair $(\cos\theta, \sin\theta)$.

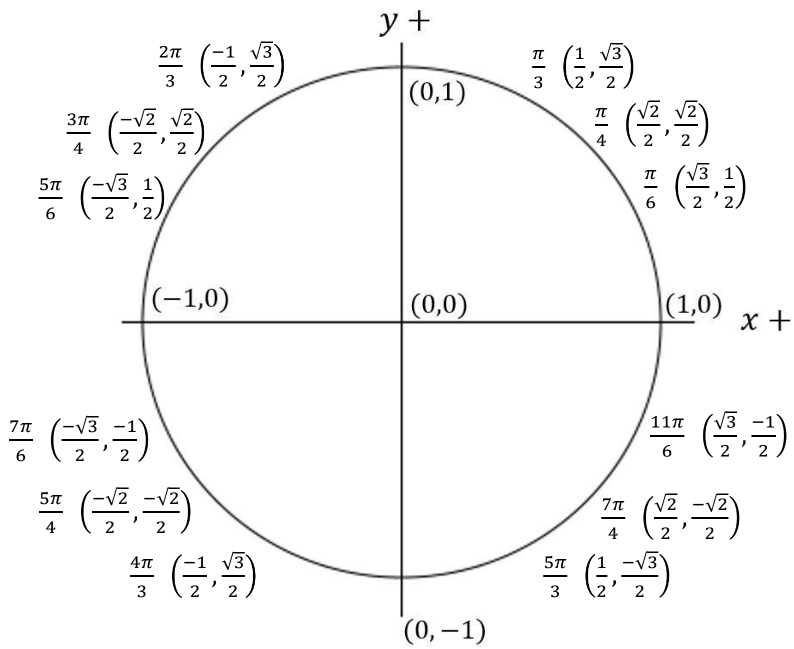


Although there are an infinite number of $(\cos\theta, \sin\theta)$ points we can create by slightly changing the value of theta, there are three of particular note that are created when we use what are widely regarded as "special" angles: $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$. These, along with the points on our circle we already know for 0 and $\frac{\pi}{2}$ result in the first quadrant of our unit circle being filled in with notable $(\cos\theta, \sin\theta)$ values.



Up until this point, we have been restricted to the angles $0-90°$ or $0-\frac{\pi}{2}$ when discussing trigonometric ratios due to the limitations imposed by the definition of a right triangle; since we have one right angle, neither of the other two angles can be greater than $90°$, or our angles would sum to a value greater than $180°$.

By redefining $\sin\theta$ and $\cos\theta$ as the height and base of our superimposed triangle measured against the vertical and horizontal axes, we can determine values of these (and all other!) trigonometric functions for angles greater than $90°$ or $\frac{\pi}{2}$, with theta defined as the angle between the positive horizontal axis and the hypotenuse of our triangle, measured counter-clockwise.

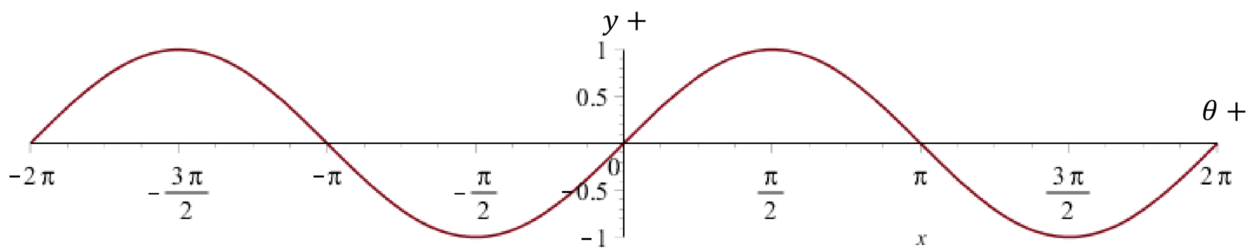


Graphing Trigonometric Functions Overview

Graphing trigonometric functions provides us with a way of visually examining how the values of trigonometric functions (particularly $\sin\theta$ and $\cos\theta$) behave as theta changes.

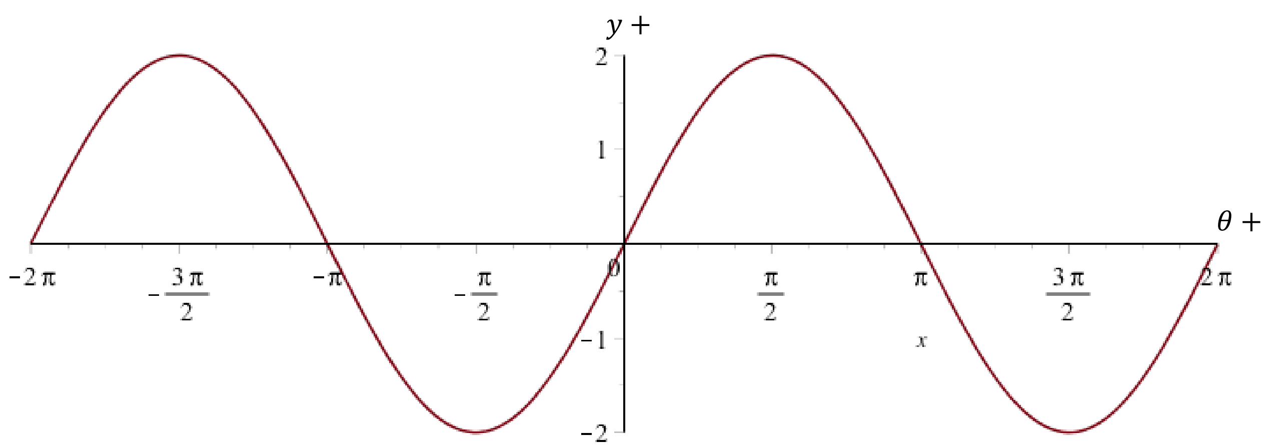
Looking at our unit circle, we can see that as theta increases from $0°$ to $90°$ or 0 to $\frac{\pi}{2}$, $\sin\theta$ (the height of our triangle) increases to 1. As theta continues to increase to $180°$ or π, $\sin\theta$ decreases to 0. At $270°$ or $\frac{3\pi}{2}$, $\sin\theta$ has decreased to -1, and after a full rotation of $360°$ or 2π, $\sin\theta$ is back to 0.

Here we have $y = \sin\theta$ graphed from -2π to 2π.

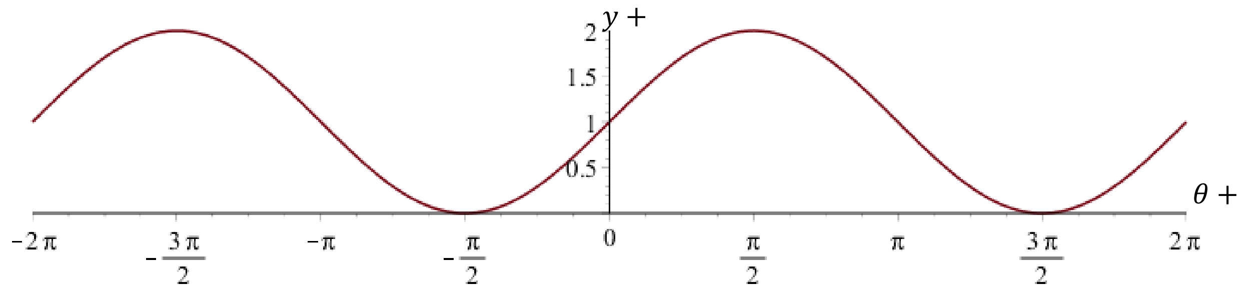


As we know, once we have completed a full rotation of the unit circle, the values we find for $\sin\theta$ begin repeating. The same is true for $\sin\theta$ values corresponding to theta values that are less than 0. This is reflected in our graph by our sine wave repeating indefinitely in both the positive and negative horizontal directions.

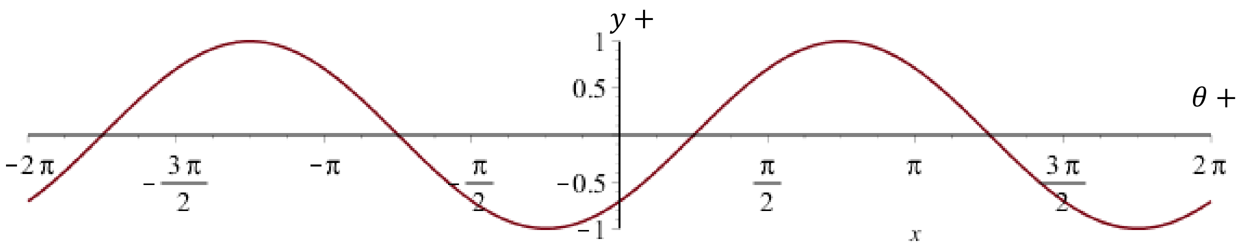
Multiplying our function by a constant will increase the **magnitude** (or distance from the horizontal axis to our maximum and minimum values) of our wave. For instance, graphing $y = 2\sin\theta$ will result in a wave that increases from 0 to 2, decreases to -2, returns to 0, then repeats.



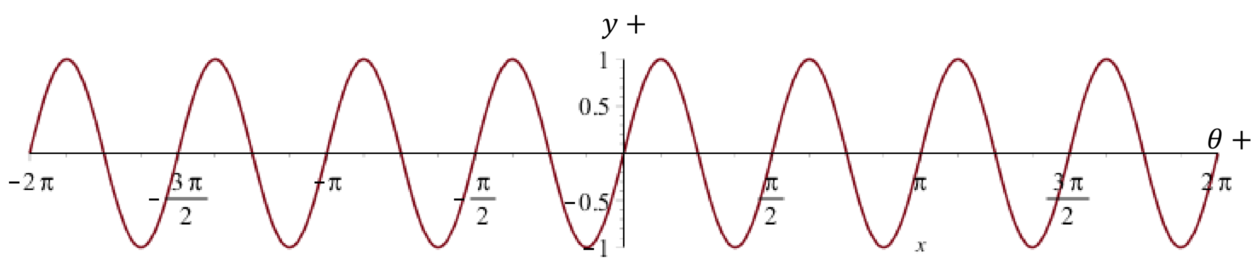
Adding a constant will shift our wave in the vertical direction. For instance, graphing $y = 1 + \sin\theta$ will result in a wave shifted 1 unit in the positive vertical direction.



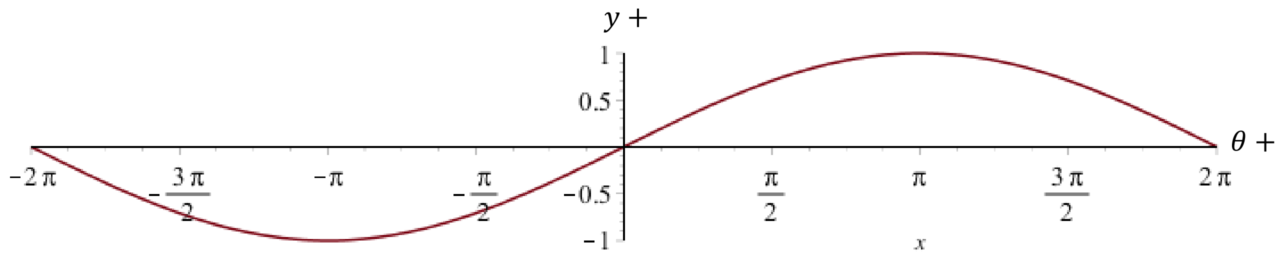
Adding a constant to our argument will shift our wave horizontally. A simple way to think of this shift is taking the constant to be your wave's new "starting point". If we add the constant $\frac{7\pi}{4}$ to our argument, $\sin(\theta + \frac{7\pi}{4})$ will be equal to $\sin\frac{7\pi}{4}$ when $\theta = 0$. Besides this horizontal shift, the wave will behave normally.



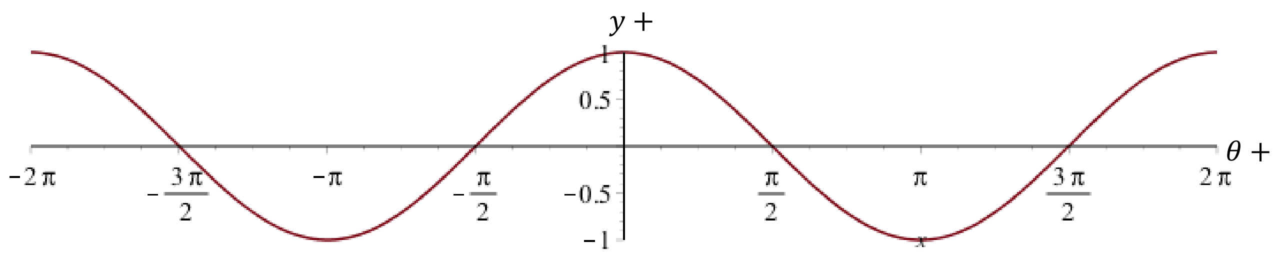
Finally, multiplying our angle by a constant will change the **wavelength** (distance from "peak-to-peak" or "trough-to-trough") of our wave. If we graph $y = \sin(4\theta)$, our resulting wave will have a wavelength that is $\frac{1}{4}$ the length of our $y = \sin\theta$ graph.



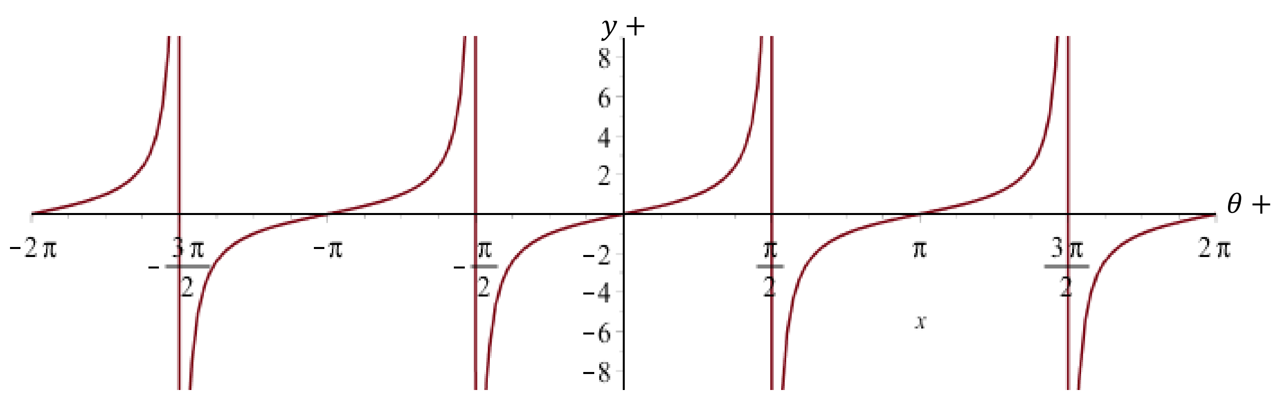
Similarly, if we graph $y = \sin\left(\frac{\theta}{2}\right)$, our resulting wave will have a wavelength 2 times the length of our $y = \sin\theta$ graph.



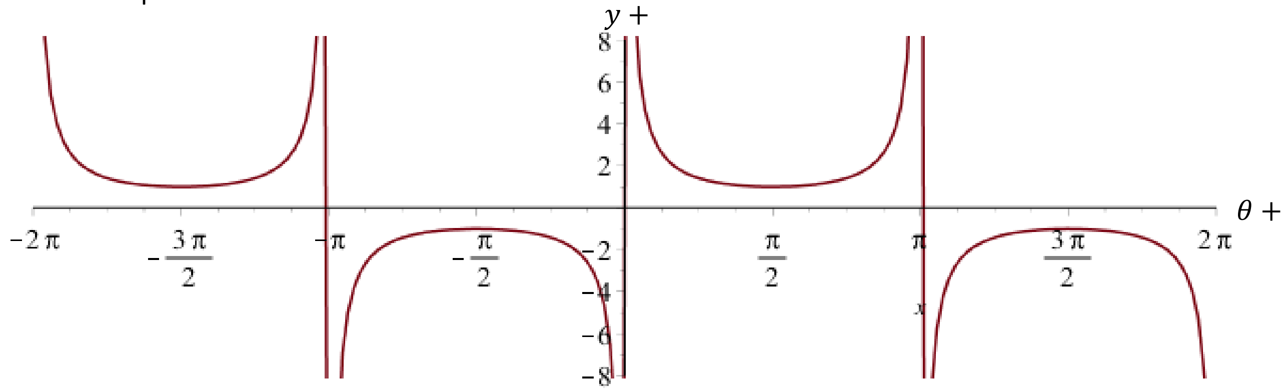
When graphed, $y = \cos\theta$ behaves exactly as $y = \sin\theta$ does, with the exception that our wave intercepts the vertical axis at $y = 1$ (since $\cos 0 = 1$).



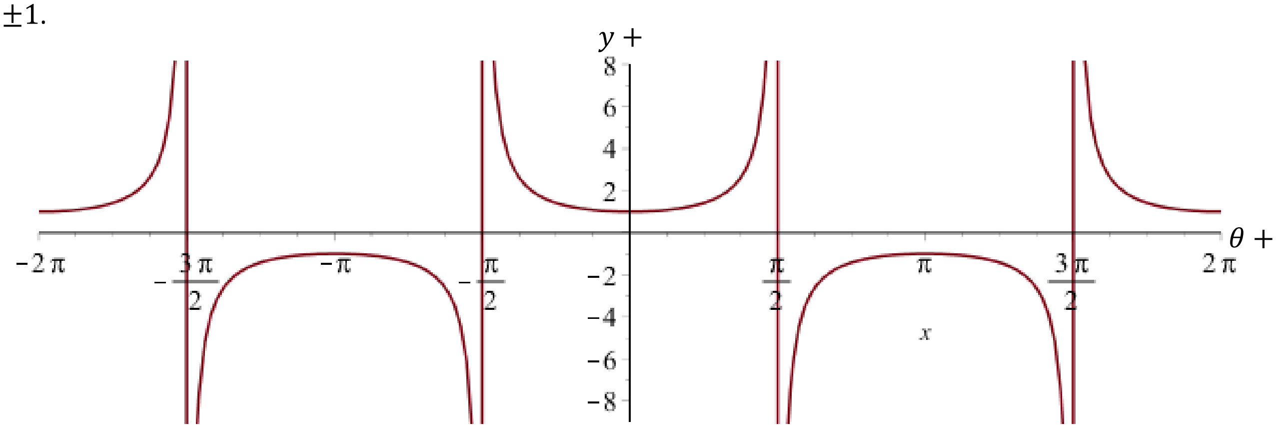
Graphing $y = \tan\theta$ results in a function with a positive slope that crosses the horizontal axis whenever $\sin\theta$ (the numerator of the tangent function) is equal to 0, and goes to $\pm\infty$ as $\cos\theta$ (the denominator of the tangent function) approaches 0. When $\cos\theta = 0, \tan\theta$ does not exist and the graph has vertical asymptotes.



Graphing the reciprocal of $\sin\theta$, $y = \csc\theta$, results in a function that approaches ∞ as $\sin\theta$ approaches 0 and is positive, $-\infty$ as $\sin\theta$ approaches 0 and is negative, has vertical asymptotes where $\sin\theta = 0$, and never crosses the θ-axis. When $sin\theta = 1$, our function is also equal to 1, and when $\sin\theta = -1$, the function is equal to -1.



Graphing $y = \sec\theta$ results in in a function that is very similar to $y = \csc\theta$, however, (since it is the reciprocal function of $\cos\theta$), it approaches $\pm\infty$ when $\cos\theta$ approaches 0, and ± 1 when $\cos\theta$ equal to ± 1.



Finally, graphing $y = \cot\theta$ results in a function very similar in behaviour to $y = \tan\theta$. It has a negative slope, crosses the horizontal axis whenever $\cos\theta$ (the numerator of the cotangent function) is equal to 0, and goes to $\pm\infty$ as $\sin\theta$ (the denominator of the cotangent function) approaches 0.