SNAP Centre Workshop

Basic Algebraic Manipulation

Simplifying Algebraic Expressions

When an expression is written in the most compact manner possible, it is considered to be **simplified**.

Not Simplified: $x(x + 4x)$

Simplified: $5x^2$

Substitution

One of the most straightforward forms of simplification arises when an expression containing variables is presented, accompanied by the variables' numerical values.

Example 1 $m + (n - 2)^2$

$m = 3, n = 5$

*By **substituting** the variables' values into the expression and evaluating, the expression can be fully simplified.*

$$(3) + \left((5) - 2\right)^2$$

$$= 3 + (3)^2$$

$$= 3 + 9$$

$$= 12$$

Note that after substitution, we follow the standard order of operations.

Grouping Like Terms

When given an expression containing variables whose values are unknown, it may still be possible to simplify the expression by **grouping like terms**. "Like terms" are terms whose variables – or combination of variables – match exactly.

When being added or subtracted in an expression, we can group like terms simply by adding or subtracting the like terms' coefficients.

Example 2 $2p + 3p$

By recognizing both terms have the variable p in common, their coefficients can be added to simplify the expression.

$$= (2 + 3)p$$

$$= \mathbf{5p}$$

The same principles apply when the variable component of the like terms is composed of more than one variable. We can add the terms' coefficients as long as the variable components of the terms match exactly.

Example 3 $8xy^2 - 17xy^2$

$= (8 - 17)xy^2$

$= \mathbf{-9xy^2}$

It is also possible for an expression to contain more than one set of like terms. To simplify, we can group all matching like terms together.

Example 4 $8q + 3q^2 - 7r + 2q + 2r$

Although it is not necessary, we can rearrange the equation to help identify and organize like terms.

$= 3q^2 + 8q + 2q - 7r + 2r$

$= 3q^2 + (8 + 2)q + (-7 + 2)r$

$= \mathbf{3q^2 + 10q - 5r}$

The q^2 and q terms do not group since they differ by an exponent and, as such, we do not consider them like terms.

Also, notice that the negative sign associated with the $7r$ stayed with its coefficient after we grouped it. If we had kept it outside the parentheses during grouping, the final expression would have incorrectly contained the term $-9r$.

Distribution

Another way we can simplify certain algebraic expressions is by **distribution**. Algebraic expressions have a distributive property whereby the term in front of a set of parentheses can be multiplied through the terms contained within the parentheses.

Example 5 $3(x + 2)$

$= (3)(x) + (3)(2)$

$= 3x + 6$

The term being distributed does not need to be a constant. It can be a combination of constants and variables. Additionally, if there is a negative sign associated with the term being distributed, it must be distributed through the parentheses along with the term.

Example 6 $-3y(8x - 2y)$

$= (-3y)(8x) + (-3y)(-2y)$

$= \mathbf{-24xy + 6y^2}$

As mentioned, the negative sign is distributed with the $3y$ term. When multiplied by the $-2y$ term, the result is a positive term, which can be seen in the final simplified expression as $6y^2$.

Simplifying an expression will often involve distribution, followed by grouping of like terms, and finally substitution. In these cases, it is important to follow the correct order of operations. Also, it can be useful to break the problem down into smaller problems to make it more manageable.

Example 7 $3x^3 - x(x^2 + x(3y - 2y + 2x))$

$Simplify\ the\ equation,\ given\ the\ following\ conditions$:

$If\ x = 4, y = 3$

$If\ x = -2, y = 5$

$If\ x = 3, y = -1$

We could substitute the given x and y values into the expression immediately, however, since three different sets of values are given, it is a better idea to perform distribution and like term grouping simplifications first.

We can begin by grouping the like terms in the innermost set of parentheses. Here, simplifying the expression $3y - 2y + 2x$ can be seen as a smaller simplification problem within the larger overall problem.

$$3x^3 - x(x^2 + x(3y - 2y + 2x))$$

$$= 3x^3 - x(x^2 + x((3 - 2)y + 2x))$$

$$= 3x^3 - x(x^2 + x(y + 2x))$$

Next, we distribute the x term across the expression in the innermost parentheses, then group the resulting like terms.

$$= 3x^3 - x(x^2 + xy + 2x^2)$$

$$= 3x^3 - x((1 + 2)x^2 + xy)$$

$$= 3x^3 - x(3x^2 + xy)$$

Distribute the $-x$ term across the remaining set of parentheses, and finish by grouping the resulting like terms.

$$= 3x^3 - 3x^3 - (x^2)y$$

$$= (3 - 3)x^3 - (x^2)y$$

$$= (0)x^3 - (x^2)y$$

$$= -(x^2)y$$

Finally, substitute the given x and y values into the simplified variable expression.

$-(x^2)y$ $\qquad$ $If\ x = 4, y = 3$

$= -(4^2)(3)$

$= -(16)(3)$

$= \mathbf{-48}$

$-(x^2)y$ $\quad$ $If\ x = -2, y = 5$

$= -((-2)^2)(5)$

$= -(4)(5)$

$= \mathbf{-20}$

$-(x^2)y$ $\quad$ $If\ x = 3, y = -1$

$= -(3^2)(-1)$

$= -(9)(-1)$

$= \mathbf{9}$

By grouping like terms and distributing first, our substitution step was made considerably easier.

We are not limited to distributing single terms across parentheses. It can also be done with polynomials.

Example 8 $\quad$ $(3x - 2y)(7x + 4y)$

We begin by distributing the first term in our first expression through the entire second expression, then distribute our second term in our first expression through the entire second expression.

$= 3x(7x + 4y) - 2y(7x + 4y)$

$= 21x^2 + 12xy - 14xy - 8y^2$

We finish by grouping all like terms.

$= \mathbf{21x^2 - 2xy - 8y^2}$

Example 9 $\quad$ $(3x^2 + 2x + 1)(x + 5)$

Again, we distribute each of the terms in our first expression across our entire second expression.

$= 3x^2(x + 5) + 2x(x + 5) + 1(x + 5)$

$= 3x^3 + 15x^2 + 2x^2 + 10x + x + 5$

Now, we group all like terms.

$= \mathbf{3x^3 + 17x^2 + 11x + 5}$

Solving "One Equation, One Unknown" Algebraic Equations

Beyond simplifying expressions, we can also use algebraic manipulation to solve equations.

It is not unusual to be presented with an equation containing some unknown variable whose value can be determined using algebraic manipulation.

When given a single equation that contains a single unknown variable, we can determine the value(s) that could be substituted into the equation in order to make it a true statement. This value, or set of values, is called the **solution**.

One way to check for solutions is to substitute a given value into an equation.

Example 10 $5x + 2 = -4 + 7x$ $\quad$ *Substitute to see if* $x = 3$ *is a solution.*

$$5(3) + 2 = -4 + 7(3)$$

$$15 + 2 = -4 + 21$$

$$17 = 17$$

Since the statement $17 = 17$ is **true**, $x = 3$ is a solution.

Example 11 $5x + 2 = -4 + 7x$ $\quad$ *Substitute to see if* $x = 2$ *is a solution.*

$$5(2) + 2 = -4 + 7(2)$$

$$10 + 2 = -4 + 14$$

$$12 = 10$$

Since the statement $12 = 10$ is **false**, we know that $x = 2$ is not a solution.

In order to avoid checking numbers one at a time to find solutions, we can use algebraic manipulation to isolate the variable in a given equation, effectively giving the solution.

Example 12 $5x + 2 = -4 + 7x$

The first step in isolating the variable x in the above equation is grouping all of the terms containing x on one side of the equation, and all of the constants on the other. This can be accomplished by performing the same operations on both sides of the equation.

$$5x + 2 - 7x - 2 = -4 + 7x - 7x - 2$$

$$5x - 7x = -4 - 2$$

To eliminate $7x$ from the right side, it was subtracted from both the right and the left sides. Similarly, to eliminate 2 from the left side, we subtracted it from both the right and the left sides.

Next, it is necessary to group the like terms on the left side of the equation.

$$(5 - 7)x = -6$$

$$-2x = -6$$

Finally, divide both sides by the unknown variable's coefficient, making sure to include the negative sign.

$$\frac{-2x}{-2} = \frac{-6}{-2}$$

$$x = 3$$

The resulting equation indicates to us that $x = 3$ *is not just a solution to the given problem (as was already shown in the previous example to be true), but the* ***only solution****, eliminating the need to substitute any other values of* x *into the equation.*

Sometimes we need to simplify an expression within an equation before finding a solution.

Example 13 $3(z + 7) - z = 14 - 2z$

First, the 3 *is distributed across the parentheses on the left hand side of the equation.*

$$3z + 21 - z = 14 - 2z$$

Next, group any like terms on both sides of the equation.

$$2z + 21 = -2z + 14$$

Once we have grouped all like terms, we can move the terms containing z *to either side of the equation, and all constants to the other.*

$$2z + 21 - 2z - 14 = -2z + 14 - 2z - 14$$

$$7 = -4z$$

Finally, we divide both sides by the variable term's coefficient.

$$\frac{7}{-4} = \frac{-4z}{-4}$$

$$-\frac{7}{4} = z$$

The solution for this particular equation is a fraction, and – unless specified otherwise – it is perfectly acceptable to leave it as one.

Special Cases

There are two special cases that we can encounter when solving algebraic equations, both of which occur when all variable terms cancel to 0.

One possible outcome is that a true statement remains once the variable terms cancel.

Example 14 $2p - 6 - 2(p - 3) + 4 = 4$

$$2p - 6 - 2p + 6 + 4 = 4$$

$$4 = 4$$

*After we simplified the left side of the equation, all terms containing the variable p cancel, leaving the true statement 4 = 4. There is no real number p can be that would make this statement false, therefore the solution is **all real numbers**.*

The second possible outcome is that a false statement remains once all variable terms cancel.

Example 15 $3(2t + 4) = 2(3t + 3) + 4$

$$6t + 12 = 6t + 6 + 4$$

$$6t + 12 = 6t + 10$$

After simplifying both sides of the equation, the next step should be to move all terms containing t to one side of the equation. Attempting to do this, however, results in all terms containing t cancelling.

$$6t + 12 - 6t = 6t + 10 - 6t$$

$$12 = 10$$

After simplification, the false statement $12 = 10$ *remains. There is no real number t that would result in this statement being true, indicating there is **no solution**.*

Isolating Variables

When presented with an equation that has more than one unknown term, it is sometimes necessary to be able to isolate one of the variables and express it in terms of the other variables.

Methods similar to those used in solving equations with one unknown are used, however, the ability to judge when to – and when not to – simplify a given term is also needed.

Example 16 $3p - 2q + 4 = \frac{5q + p}{3}$ $\quad$ $Express\ p\ in\ terms\ of\ q.$

The goal here is to move all constants and terms containing q to one side of the equation, and isolate p on the other side of the equation.

Simplifying both sides of the equation seems like a good place to start, however, distributing the 3 in the denominator on the right side of the equation would result in the need to work with fractions. Instead, notice that multiplying both sides of the equation by 3 results in whole numbers throughout the entire equation.

$$3(3p - 2q + 4) = \frac{3(5q + p)}{3}$$

$$9p - 6q + 12 = 5q + p$$

Next, perform the appropriate addition and subtraction operations to gather all p terms on the left, and all constants and q terms on the right, and then group any like terms.

$$9p - 6q + 12 - p + 6q - 12 = 5q + p - p + 6q - 12$$

$$9p - p = 5q + 6q - 12$$

$$8p = 11q - 12$$

Finally, divide both sides of the equation by 8 *to isolate* p *completely.*

$$\frac{8p}{8} = \frac{11q - 12}{8}$$

$$p = \frac{11q - 12}{8}$$

The next example illustrates how an expression inside parentheses can – and usually should – be treated in the same way a single term would be treated as long as **none of the terms inside the parentheses contain the variable that is being isolated**.

Example 17 $y + 2 = 2x(y + 3)$ $\quad$ $Express\ x\ in\ terms\ of\ y.$

Normally, our first step in this problem would be to distribute $2x$ *across the parentheses on the right side of the equation, however, once this term is distributed, isolating* x *would be far more difficult. Instead, both sides of the equation will be* ***divided by the term in parentheses.***

$$\frac{y + 2}{y + 3} = \frac{2x(y + 3)}{y + 3}$$

$$\frac{y + 2}{y + 3} = 2x$$

$\frac{y+3}{y+3}$ *simplifies to* 1*, leaving* $2x$ *on the right side of the equation. As in previous examples, the only step left at this point is to divide by the* x *term's constant coefficient.*

$$\frac{y + 2}{2(y + 3)} = x$$

$$OR$$

$$\frac{y + 2}{2y + 6} = x$$

Now x *is being expressed in terms of* y*. Note that the* 2 *in the denominator on the left side of the equation can be distributed, or remain outside the parentheses.*

Often, there is more than one way to manipulate an equation algebraically. The following example can be solved using two equally valid methods.

Example 18 $\frac{-4nm + 4}{n} = 2m$ $\quad$ $Express\ m\ in\ terms\ of\ n.$

The first method involves distributing the n *in the denominator on the left side of the equation across the terms in parentheses.*

$$\frac{-4nm}{n} + \frac{4}{n} = 2m$$

$$-4m + \frac{4}{n} = 2m$$

Next, we will add $4m$ to both sides of the equation, group like terms, then divide both sides of the equation by the m term's constant coefficient to isolate m completely.

$$-4m + \frac{4}{n} + 4m = 2m + 4m$$

$$\frac{4}{n} = (2 + 4)m$$

$$\frac{4}{n} = 6m$$

$$\frac{4}{6n} = \frac{6m}{6}$$

$$\frac{4}{6n} = m$$

$$\frac{2}{3n} = m$$

The second method begins by multiplying both sides of the equation by n.

Example 19 $\frac{-4nm + 4}{n} = 2m$ $\quad$ $Express\ m\ in\ terms\ of\ n.$

$$\frac{n(-4nm + 4)}{n} = n(2m)$$

$$-4nm + 4 = 2nm$$

Next, all terms containing m are moved to one side of the equation, after which like terms are grouped.

$$-4nm + 4 + 4nm = 2nm + 4nm$$

$$4 = (2 + 4)nm$$

$$4 = 6nm$$

Finally, we divide both sides of the equation by the constant coefficient ***and*** *the variable coefficient associated with the m term, and simplify.*

$$\frac{4}{6n} = \frac{6nm}{6n}$$

$$\frac{2}{3n} = m$$